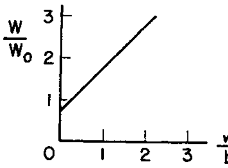

Fig. 14 Load-displacement of point of attachment after 10% cable strain.

$$\frac{w}{b_0} = \frac{2}{\phi^2}(1 + \theta \cosh\phi) - 1 = 2.22, 0.61, 0.07$$

Figure 14 shows the generalized load vs displacement curve obtained by plotting corresponding values of W/W_0 vs w/b_0. Figure 15 shows the original shape of the catenary curtain and its shape after straining, for the three loading conditions calculated. These curves are plotted from the relations

$$\frac{y_0}{b_0} = \left(\frac{x_0}{a_0}\right)^2 = \left(\frac{x}{a}\right)^2$$

$$\frac{y}{b_0} = (\theta + 1)\left(\frac{x}{a}\right)^2 + \frac{\phi^2\theta}{12}\left(\frac{x}{a}\right)^4 + \frac{\phi^4\theta}{360}\left(\frac{x}{a}\right)^6 \cdots$$

$$\frac{u_0}{b_0} = 1$$

$$\frac{u_{(x=0)}}{b_0} = \frac{2}{\phi^2}\frac{b_1}{b_0} = 2.79, 1.19, 0.66$$

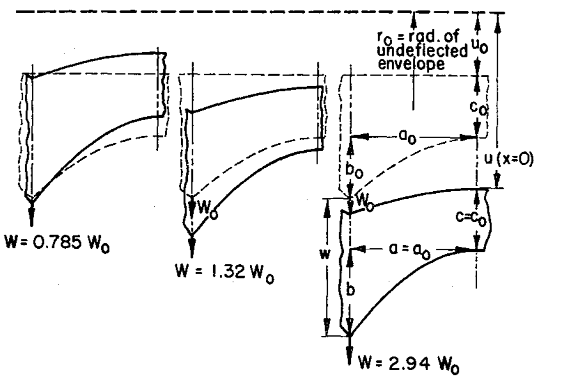

Fig. 15 Change in shape due to 10% cable strain.

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[1] Donnell, L. H., "Stress concentration at the ends of long reinforcements," Natl. Geograph. Soc. Stratosphere Ser. 2, 245 (1936).

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AUGUST 1963 AIAA JOURNAL VOL. 1, NO. 8

Case-Bounded Elastic-Plastic and Nonlinear Elastic Hollow Cylinders

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An elastically case-bonded hollow cylinder of infinite length, the mechanical response of which is perfectly elastic-plastic or nonlinearly elastic, is considered. For a perfectly elastic-plastic cylinder with the Tresca yield condition and its associated flow rule, W. T. Koiter's solution is extended to the problem of the cylinder contained in an elastic shell; the effect of the shell on the stresses is demonstrated with numerical examples. The same problem is solved for a nonlinear elastic cylinder, the second invariants J and I of the stress and strain deviator of which are assumed to have a relation $J = 4(G - gI)_2I$, G and g being material constants; incompressibility is introduced to make the analysis simple. The critical state is defined in such a way that failure occurs at the point where I reaches a critical value I_{cr}. Comparison of the stresses in the elastic-plastic and in the nonlinear elastic cylinder for the inner surface of the cylinder reaching the critical condition shows little difference, at least for the specific values of parameters chosen for computation.

1. Introduction

The present investigation deals with an ideally elastic-plastic and a nonlinearly elastic hollow cylinder of infinite length enclosed in an elastic shell and subject to an internal pressure. The nonlinearities assumed are intended to reflect approximately the mechanical response of the solid propellant material.

In the first part, the plane strain problem of an elastically compressible and plastically incompressible elastic-plastic hollow cylinder contained in an elastic shell subject to an internal pressure is considered. Little attention has been paid to this problem so far, although the same problem without elastic shell has been solved under various yield conditions and stress-strain relations.

For a perfectly plastic material, the component ϵ_{ij} of the (total) strain can be written as the sum of the components of the elastic and the plastic strains, ϵ_{ij}^E and ϵ_{ij}^P:

$$\epsilon_{ij} = \epsilon_{ij}^E + \epsilon_{ij}^P \tag{1}$$

The incompressibility of the plastic deformation

$$\epsilon_{kk}^P = 0 \tag{2}$$

is assumed; the usual convention of summation over repeated subscripts is adopted.

Received February 18, 1963. This research was supported by the Office of Naval Research under Contract Nonr 266(78).

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ϵ_{ij}^E is related to the stresses in accordance with the elastic stress-strain relation:

$$\sigma_{kk} = 3K\epsilon_{kk}^E = 3K\epsilon_{kk} \qquad s_{ij} = 2Ge_{ij}^E \tag{3}$$

or, in case the material is also elastically incompressible,

$$\epsilon_{kk}^E = \epsilon_{kk} = 0 \qquad s_{ij} = 2Ge_{ij}^E \tag{3'}$$

where σ_{ij}, s_{ij}, and e_{ij}^E denote the components of the stress tensor, of the stress deviator, and of the elastic strain deviator, respectively; K is the bulk modulus, and G is the shear modulus.

The components of the plastic strain rate, associated with the yield condition $f(\sigma_{ij}) = 0$, are given by

$$\begin{aligned} \dot{\epsilon}_{ij}^P = \dot{e}_{ij}^P = \lambda[\partial f(\sigma_{ij})/\partial\sigma_{ij}] \text{ if } f(\sigma_{ij}) = 0 \text{ and } \dot{f}(\sigma_{ij}) = 0 \\ \text{and } \dot{\epsilon}_{ij}^P = 0 \text{ if } f(\sigma_{ij}) < 0 \text{ or if } f(\sigma_{ij}) = 0 \text{ and } \dot{f}(\sigma_{ij}) < 0 \end{aligned} \tag{4}$$

where λ is a positive parameter and dots denote differentiation with respect to some parameter that is a monotonically increasing function of time.

For the Tresca yield condition, $f(\sigma_{ij})$ is of the form[1]

$$f(\sigma_{ij}) = f_{1,4}\cdot f_{2,5}\cdot f_{3,6} = 0$$

with

$$\begin{aligned} 4f_{1,4} &= [(\sigma_2 - \sigma_3)^2 - 4k^2] \\ 4f_{2,5} &= [(\sigma_3 - \sigma_1)^2 - 4k^2] \\ 4f_{3,6} &= [(\sigma_1 - \sigma_2)^2 - 4k^2] \end{aligned} \tag{5}$$

where σ_1, σ_2, and σ_3 are the principal stresses, and k is the yield stress in pure shear.

The following relations associated with the Tresca yield condition therefore are obtained from Eqs. (4) and (5):

$$\begin{aligned} &\dot{\epsilon}_1^P : \dot{\epsilon}_2^P : \dot{\epsilon}_3^P = 1:0:-1 \text{ if } \sigma_1 - \sigma_3 = 2k \text{ and } \dot{\sigma}_1 - \dot{\sigma}_3 = 0 \\ &\dot{\epsilon}_1^P = \dot{\epsilon}_2^P = \dot{\epsilon}_3^P = 0 \text{ if } \sigma_1 - \sigma_3 = 2k \text{ and } \dot{\sigma}_1 - \dot{\sigma}_3 < 0 \\ &\qquad \text{of if } \sigma_1 - \sigma_3 < 2k \end{aligned} \tag{6}$$

where it is assumed that $\sigma_1 > \sigma_2 > \sigma_3$.

Koiter[2] applied Eqs. (1–3 and 6) to the problem of an infinite elastic-plastic hollow cylinder (plane strain) under internal pressure, obtaining a closed-form solution of considerable simplicity. In the first part of the present investigation, the same assumptions are shown to produce a closed-form solution to the plane strain problem of an elastic-plastic hollow cylinder contained in an elastic shell subject to internal pressure; the effect of the elastic shell on the stresses and strains in the cylinder is discussed with the aid of numerical examples. In the second part, the incompressible elastic cylinder of more general nonlinearity with respect to the deviatoric stress and strain relation (referred to as nonlinear elastic)

$$\epsilon_{kk} = 0 \qquad s_{ij} = 2(G - gI^n)e_{ij} \tag{7}$$

is discussed under the assumption that the material fails when I reaches a critical value I_{cr}, where I is the second invariant of strain deviation

$$I = \tfrac{1}{2}\, e_{ij}e_{ij} \tag{8}$$

and G and g are material constants.

2. Elastic-Plastic Solution

The equation of equilibrium

$$r\sigma_{r,r} + \sigma_r - \sigma_\theta = 0 \tag{9}$$

is satisfied automatically if the stress function ϕ is introduced so that $\sigma_r = \phi/r$ and $\sigma_\theta = \phi_{,r}$, where σ_r and σ_θ are the radial and tangential stresses in a cylindrical coordinate system, and $\sigma_{r,r}$ and $\phi_{,r}$ indicate differentiations of σ_r and ϕ with respect to the radius r, respectively. (This convention is used throughout.)

In the region where the yield condition $f(\sigma_{ij}) = 0$ has not been attained, the plastic strain $\epsilon_{ij}^P = 0$. The radial displacement u therefore is related to the stresses in the form

$$\begin{aligned} Eu_{,r} &= (1 - \nu^2)\sigma_r - \nu(1 + \nu)\sigma_\theta = \\ &\qquad (1 - \nu^2)\phi/r - \nu(1 + \nu)\phi_{,r} \\ Eu/r &= (1 - \nu^2)\sigma_\theta - \nu(1 + \nu)\sigma_r = \\ &\qquad (1 - \nu^2)\phi_{,r} - \nu(1 + \nu)\phi/r \end{aligned} \tag{10}$$

considering the strain-displacement relation

$$\epsilon_r = u_{,r} \qquad \epsilon_\theta = u/r \tag{11}$$

and the elastic stress-strain relations in their usual form, where ϵ_r and ϵ_θ are the (total) radial and tangential strain, respectively, E is the Young's modulus, and ν is the Poisson ratio.

The displacement u is eliminated from Eqs. (10), resulting in the differential equation for the stress function

$$r\phi_{,rr} + \phi_{,r} - \phi/r = 0 \tag{12}$$

which has the solution $\phi = Ar + B/r$, where A and B are arbitrary constants. Hence the stresses are

$$\sigma_r = A + B/r^2 \qquad \sigma_\theta = A - B/r^2 \qquad \sigma_z = 2\nu A \tag{13}$$

where σ_z is the axial stress.

The equilibrium of the elastic shell requires the relation

$$\sigma_r]_{r=b} = -(h/b)\sigma_c \tag{14}$$

between the shell stress σ_c and the radial stress in the cylinder $\sigma_r]_{r=b}$ at the interface between the cylinder and the shell, while σ_c is in turn related to the strain of the cylinder $\epsilon_\theta]_{r=b}$ at this interface by the relation

$$\sigma_c = \{E_c/(1 - \nu_c^2)\}\epsilon_\theta]_{r=b} \tag{15}$$

since $\epsilon_\theta]_{r=b}$ is identical to the shell strain ϵ_c; h is the thickness of the shell, b the outer radius of the hollow cylinder, and E_c and ν_c are Young's modulus and the Poisson ratio of the shell, respectively.

The material remains elastic in the region $r > r_0$ and is plastic in the region $r \leqq r_0$, where r_0 denotes the position of the elastic-plastic boundary.

The Tresca yield condition at the boundary $r = r_0$,

$$\sigma_\theta - \sigma_r = 2k \tag{16}$$

when the inequality

$$\sigma_\theta > \sigma_z > \sigma_r \tag{17}$$

is assumed.

Equations (14) and (16) are used to determine two constants A and B in Eq. (13) in terms of the unknown σ_c:

$$A = (kr_0^2 - bh\sigma_c)/b^2 \qquad B = -kr_0^2$$

The strains and stresses therefore can be expressed in terms of σ_c with the aid of Eqs. (11) and (13). The use of Eq. (15) then determines σ_c:

$$\sigma_c = k(E_c'/2G)(r_0/b)^2(1/\alpha) \tag{18}$$

where $E_c' = E_c/(1 - \nu_c^2)$ and $\alpha = [1 + (1 - 2\nu)\mu]/2(1 - \nu)$; $\mu = E_c'h/2Gb$ reflects the strength of the elastic shell. Hence the solution has been obtained for the elastic part of the cylinder as given in column III of Table 1.

In the plastic region where Eq. (16) is valid, Eq. (9) can be integrated easily:

$$\sigma_r = 2k \ln r + C$$

The integration constant C is determined from the continuity of σ_r at the elastic-plastic boundary:

$$C = k[(r_0^2/b^2) - 1] - 2k \ln r_0 - (h/b)\sigma_c$$

The relation between the axial strain ϵ_z and stresses is always

Table 1 Solution to the ideal elastic-plastic cylinder

	PLASTIC PART $a \le r \le r_0$		III ELASTIC PART $r_0 \le r \le b$
	I INNER PLASTIC PART $a \le r \le r_1$	II OUTER PLASTIC PART $r_1 \le r \le r_0$	
$\frac{2G\epsilon_r}{k}$	$-2(1-\nu)\left(\frac{r_0}{r}\right)^2 - \frac{(1-2\nu)^2}{2(1+\nu)}\left(\frac{r_1}{r}\right)^2 + \frac{3(1-2\nu)}{2(1+\nu)}\left[\ln\left(\frac{r}{r_0}\right)^2 + \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2 + \frac{4}{3}\right]$	$-2(1-\nu)\left(\frac{r_0}{r}\right)^2 + (1-2\nu)\left[\ln\left(\frac{r}{r_0}\right)^2 + \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2 + 1\right]$	$-\left(\frac{r_0}{r}\right)^2 + (1-2\nu)\left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2$
$\frac{2G\epsilon_\theta}{k}$	$2(1-\nu)\left(\frac{r_0}{r}\right)^2 + \frac{(1-2\nu)^2}{2(1+\nu)}\left(\frac{r_1}{r}\right)^2 + \frac{3(1-2\nu)}{2(1+\nu)}\left[\ln\left(\frac{r}{r_0}\right)^2 + \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2 - \frac{2}{3}\right]$	$2(1-\nu)\left(\frac{r_0}{r}\right)^2 + (1-2\nu)\left[\ln\left(\frac{r}{r_0}\right)^2 + \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2 - 1\right]$	$\left(\frac{r_0}{r}\right)^2 + (1-2\nu)\left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2$
$\frac{\sigma_r}{k}$	$\ln\left(\frac{r}{r_0}\right)^2 + \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2 - 1$	$\ln\left(\frac{r}{r_0}\right)^2 + \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2 - 1$	$-\left(\frac{r_0}{r}\right)^2 + \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2$
$\frac{\sigma_\theta}{k}$	$\ln\left(\frac{r}{r_0}\right)^2 + \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2 + 1$	$\ln\left(\frac{r}{r_0}\right)^2 + \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2 + 1$	$\left(\frac{r_0}{r}\right)^2 + \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2$
$\frac{\sigma_z}{k}$	$\ln\left(\frac{r}{r_0}\right)^2 + \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2 + 1$	$2\nu\left[\ln\left(\frac{r}{r_0}\right)^2 + \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2\right]$	$2\nu\left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2$
$p - r_0$ RELATION	$\frac{p}{k} = 1 + \ln\left(\frac{r_0}{a}\right)^2 - \left(1-\frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2$		$\frac{\sigma_c}{k} = \frac{E_c'}{2G\alpha}\left(\frac{r_0}{b}\right)^2$, $\mu = \frac{E_c' h}{2G b}$
$r_1 - r_0$ RELATION	$\left(1-\frac{\mu}{\alpha}\right)(1-2\nu)\left(\frac{r_0}{b}\right)^2 + (1-2\nu)\ln\left(\frac{r}{r_0}\right)^2 + 1 = 0$		$\alpha = \frac{1+(1-2\nu)\mu}{2(1-\nu)}$, $E_c' = \frac{E_c}{1-\nu_c^2}$
THE CONDITION UNDER WHICH THE INNER PLASTIC REGION APPEARS BEFORE r_0 REACHES THE OUTER BOUNDRY			$\ln\rho^2 \ge \frac{1}{1-2\nu} + 1 - \frac{\mu}{\alpha}$, $\rho = \frac{b}{a}$

elastic because the plastic component of the axial strain $\epsilon_z^P = 0$ even in the plastic region, since $\epsilon_z^P = 0$ when the plastic state of stress has been attained, and $\dot{\epsilon}_z^P = 0$ from Eq. (6). Hence, the third of Eqs. (13) also is valid in the plastic region, and $\sigma_z = \nu(\sigma_r + \sigma_\theta)$. The stresses in the plastic region are shown in column II of Table 1.

The relation between the position of the elastic-plastic boundary r_0 and the internal pressure p is obtained from the fact that σ_r at the inner surface ($r = a$) must be equal to p:

$$p/k = 1 + \ln(r_0/a)^2 - [1 - (\mu/\alpha)](r_0/b)^2 \tag{19}$$

The first of Eqs. (3) written in the form

$$u_{,r} + \frac{u}{r} = \frac{1}{3K}(\sigma_r + \sigma_\theta + \sigma_z) = \frac{1 - 2\nu}{2G}(\sigma_r + \sigma_\theta) \tag{20}$$

can be integrated for u after the expressions in column II of Table 1 are substituted for σ_r and σ_θ:

$$u = \frac{D}{r} + \frac{k(1-2\nu)}{G} r\left[\ln\left(\frac{r}{r_0}\right) + \frac{1}{2}\left(\frac{r_0^2}{b^2} - \frac{c}{b}\frac{\sigma_c}{k} - 1\right)\right]$$

where σ_c is given in Eq. (18). The integration constant D is determined from the continuity of u at the elastic-plastic boundary:

$$D = k(1 - \nu)r_0^2/G$$

The strains obtained from Eqs. (11) are shown in column II of Table 1.

Since $0 < \nu < 0.5$, it can be shown that σ_z is the intermediate stress in the elastic region, whereas the stresses given in column II of Table 1 are consistent with the assumption Eq. (17) only for $r > r_1$, where r_1 is the solution of the transcendental equation

$$\left(1 - \frac{\mu}{\alpha}\right)(1 - 2\nu)\left(\frac{r_0}{b}\right)^2 + (1 - 2\nu)\ln\left(\frac{r_1}{r_0}\right)^2 + 1 = 0 \tag{21}$$

σ_r is always the smallest stress.

Hence, Eq. (17) holds in the region $r_1 < r < r_0$ (referred to as the *outer plastic region*), whereas the relation

$$\sigma_\theta = \sigma_z > \sigma_r \qquad \dot{\sigma}_\theta = \dot{\sigma}_z = \dot{\sigma}_r \tag{22}$$

is valid in the region $a < r < r_1$ (referred to as the *inner plastic region*) as long as

$$\dot{\epsilon}_z^P > 0 \qquad \dot{\epsilon}_\theta^P > 0 \qquad \dot{r}_1 > 0 \tag{23}$$

with increasing internal pressure ($\dot{p} > 0$).

The arguments used by Koiter in the original derivation[2] of Eqs. (22) for the cylinder without elastic shell are valid also when the cylinder has an elastic outside shell.

It now can be shown from the condition for the nonexistence of real roots of Eq. (21) that if

$$\ln\rho^2 < [1/(1 - 2\nu)] + 1 - (\mu/\alpha) \tag{24}$$

where $\rho = b/a$, the inner plastic region never will appear.

Equation (24) shows that the existence of the elastic shell produces the inner plastic region in thinner cylinders, in which, without the elastic shell, it never would appear.

The stresses in the inner plastic region are the same as in the outer region except for σ_z, which now is equal to σ_θ, as shown in column I of Table 1.

The first two members of Eq. (20) can be used to determine the displacement u:

$$u = \frac{D'}{r} + \frac{(1 - 2\nu)k}{2(1 + \nu)G} \cdot r \cdot \left[3\ln\left(\frac{r}{r_0}\right) + \frac{3}{2}\left(1 - \frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2 - 1\right]$$

where the integration constant D' is obtained from continuity of displacement at $r = r_1$:

$$D' = (1 - \nu)\frac{kr_0^2}{G} + \frac{(1 - 2\nu)^2}{4(1 + \nu)} \cdot \frac{kr_1^2}{G}$$

The strains are obtained from Eqs. (11) and are shown in column I of Table 1.

From differentiation of Eq. (19) with respect to time parameter, $\dot{r}_0$ can be proved to be positive if $\dot{p} > 0$:

$$\dot{r}_0 = \frac{\dot{p}}{2k} \cdot \frac{r_0}{1 - [1 - (\mu/\alpha)](r_0^2/b^2)} > 0$$

Therefore, $\dot{r}_1$ can be seen to be positive from differentiation of Eq. (21) if $\dot{p} > 0$:

$$\dot{r}_1 = \dot{r}_0\{(r_1/r_0) - [1 - (\mu/\alpha)](r_0r_1/b^2)\} > 0$$

The differentiation of the stress in the inner plastic region with respect to time parameter produces the stress rates

$$\dot{\sigma}_r = \dot{\sigma}_\theta = \dot{\sigma}_z = -2k(\dot{r}_0/r_0)\{1 - [1 - (\mu/\alpha)](r_0/b)^2\}$$

which, in turn, give the elastic strain rates

$$\dot{\epsilon}_r^E = \dot{\epsilon}_\theta^E = \dot{\epsilon}_z^E = -\frac{k(1 - 2\nu)\dot{r}_0}{G(1 + \nu)r_0}\left[1 - \left(1 - \frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2\right]$$

since

$$\dot{\epsilon}_r^E = \dot{\epsilon}_z^E = \dot{\epsilon}_\theta^E = (1/E)[\dot{\sigma}_\theta - \nu(\dot{\sigma}_z + \dot{\sigma}_r)]$$

Hence

$$\dot{\epsilon}_\theta^P = \dot{\epsilon}_\theta - \dot{\epsilon}_\theta^E = 2(1 - \nu)\frac{kr_0\dot{r}_0}{Gr^2} - \frac{k(1 - 2\nu)\dot{r}_0}{2G(1 + \nu)r_0} \times \left[1 - \left(1 - \frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2\right]\left[1 - (1 - 2\nu)\left(\frac{r_1}{r}\right)^2\right] > 0$$

$$\dot{\epsilon}_z^P = -\dot{\epsilon}_z^E = \frac{k(1 - 2\nu)\dot{r}_0}{G(1 + \nu)r_0}\left[1 - \left(1 - \frac{\mu}{\alpha}\right)\left(\frac{r_0}{b}\right)^2\right] > 0$$

where $\dot{\epsilon}_\theta$ is computed from ϵ_θ in column I of Table 1.

The solution for the inner plastic region given in column I of Table 1 is correct, since the conditions in Eqs. (23) all are satisfied.

Table 1 therefore shows the complete solution to the plane strain problem of the elastic-plastic hollow cylinder with elastic shell. In addition to the stresses and strains, the $p \sim r_0$ and $r_1 \sim r_0$ relations, the condition under which the inner plastic region will appear and the shell stress σ_c also are given there.

It should be noted that for $\mu = 0$ the solution is identical to the one given by Koiter, and the incompressible solution

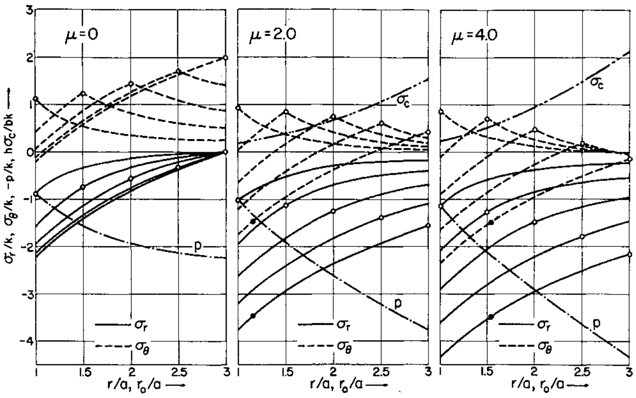


Fig. 1 Space distributions of stresses and the internal pressure p and the shell stress σ_c as function of r_0/a ($\nu = 0.3$, $\rho = 3.0$).

is obtained for $\nu = 0.5$, where the inner plastic region never appears.

Numerical examples are given for various combinations of Poisson ratio ($\nu = 0.3$ and 0.5), the strength of the elastic shell ($\mu = E_c'h/2Gb = 0$, 2.0, and 4.0), and the ratio between the outer and the inner radius ($\rho = b/a = 3.0$).

The results are shown in Figs. 1–4, where the stresses and the strains for the elastic-plastic boundary at the inner surface, at quarter points of the thickness of the cylinder and at the interface between the cylinder and shell, are plotted against r/a (the space distributions of the stresses and the strains), whereas the internal pressure p and the shell stress σ_c are plotted against r_0/a; values of p and σ_c are plotted as the ordinate when the elastic-plastic boundary occupies the position given by the abscissa. The solid circles indicate the interface between the inner and the outer plastic region, whereas open ones show the position of elastic-plastic boundaries.

The computation is carried out with the aid of a digital computer IBM 1620 not because of mathematical difficulty but because of the amount of computation involved and the frequent necessity of obtaining roots of the transcendental equations [Eqs. (19) and (21)].

3. Nonlinear Elastic Solution

In this section the hollow cylinder with isotropic stress-strain relation given by Eqs. (7) and $n = 1$ is considered at the critical instant when the inner surface starts to undergo the critical strain. Then, the stresses and strains of the ideal elastic-plastic cylinder at the same critical instant are

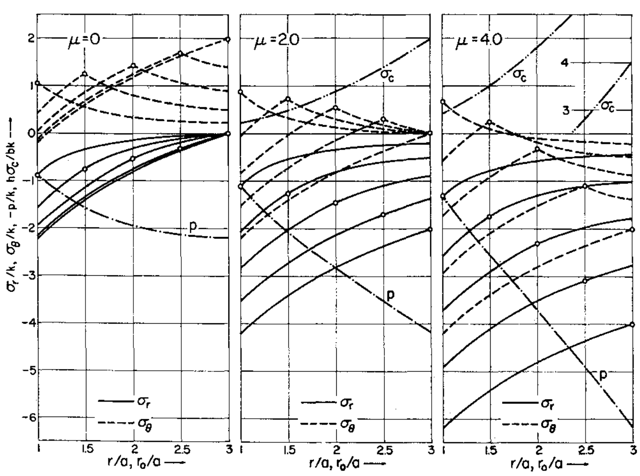


Fig. 2 Space distributions of stresses and the internal pressure p and the shell stress σ_c as function of r_0/a ($\nu = 0.5$, $\rho = 3.0$).

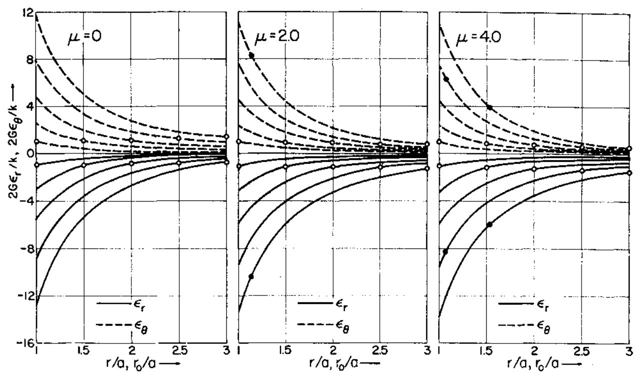


Fig. 3 Space distributions of strains ($\nu = 0.3$, $\rho = 3.0$).

obtained from Table 1, and the difference between these two solutions is illustrated by a numerical example.

Since incompressibility is assumed, it can be shown easily that

$$\epsilon_\theta = e_\theta = -\epsilon_r = -\epsilon_r = A'/r^2$$

and therefore

$$I = A'^2/r^4$$

where A' is a constant and determined by the fact that $I = I_{cr}$ at $r = a$ at the critical instant

$$A' = a^2 I_{cr}^{1/2}$$

Hence,

$$\epsilon_\theta = -\epsilon_r = I_{cr}^{1/2}\,(a/r)^2 \tag{25}$$

and, since $I = I_{cr}(a/r)^4$,

$$\begin{aligned}\sigma_r &= -2[G - gI_{cr}(a/r)^4]I_{cr}^{1/2}\,(a/r)^2 + \sigma\\ \sigma_\theta &= 2[G - gI_{cr}(a/r)^4]I_{cr}^{1/2}\,(a/r)^2 + \sigma\end{aligned} \tag{26}$$

where σ is the mean stress, $\sigma = \sigma_{kk}/3$.

When the square of both sides is taken, the second of Eqs. (7) becomes

$$J = 4(G - gI)^2 I \tag{27}$$

It is postulated further that, as shown in Fig. 5, when I reaches I_{cr}, J assumes a maximum that is written as k^2 for later comparison of the nonlinear elastic solution with the ideal elastic-plastic solution:

$$\left.\frac{\partial J}{\partial I}\right]_{I = I_{cr}} = 4(G - gI_{cr})(G - 3gI_{cr}) = 0$$

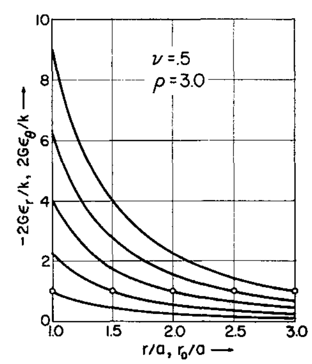
Hence,

$$I_{cr} = G/(3g)$$

and

$$k^2 = \tfrac{16}{27}\,G^2(G/g) = \tfrac{16}{9}G^2 I_{cr} \tag{28}$$

Fig. 4 Space distributions of strains ($\nu = 0.5$, $\rho = 3.0$).



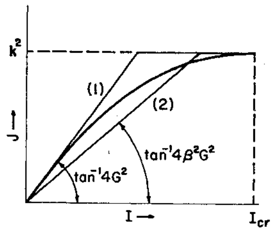


Fig. 5 Nonlinear elastic relation.

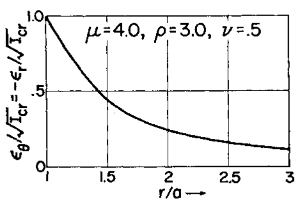


Fig. 6 Critical space distribution of strains (μ = 4.0, ρ = 3.0, ν = 0.5).

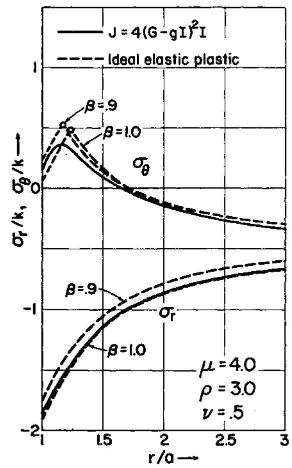


Fig. 7 Critical space distribution of stresses (μ = 4.0, ρ = 3.0, ν = 0.5).

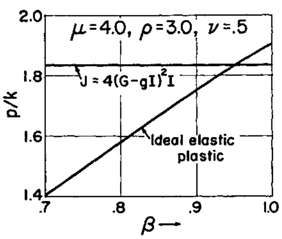


Fig. 8 Critical internal pressure as a function of β (μ = 4.0, ρ = 3.0, ν = 0.5).

The strains [Eq. (25)] now can be written in the form

$$2G\epsilon_\theta/k = -2G\epsilon_r/k = \tfrac{3}{2}(a/r)^2 \tag{29}$$

The use of Eq. (9) together with Eqs. (26) determines σ up to a constant, which in turn is determined by the boundary conditions Eqs. (14) and (15):

$$\frac{\sigma_r}{k} = -\frac{3}{2}\left[\left(\frac{a}{r}\right)^2 - (1-\mu)\left(\frac{a}{b}\right)^2 - \frac{1}{9}\left(\frac{a}{r}\right)^6 + \frac{1}{9}\left(\frac{a}{b}\right)^6\right]$$

$$\frac{\sigma_\theta}{k} = \frac{3}{2}\left[\left(\frac{a}{r}\right)^2 + (1-\mu)\left(\frac{a}{b}\right)^2 - \frac{5}{9}\left(\frac{a}{r}\right)^6 - \frac{1}{9}\left(\frac{a}{b}\right)^6\right] \tag{30}$$

The critical internal pressure is

$$p/k = \tfrac{3}{2}[1 - (1-\mu)(a/b)^2 - \tfrac{1}{9} + \tfrac{1}{9}(a/b)^4] \tag{31}$$

As an example, the strains and stresses are computed with the aid of Eqs. (29) and (30) for the case where $\mu = 4.0$ and $\rho = 3.0$ and are shown in Figs. 6 and 7.

For comparison, the ideal elastic-plastic incompressible cylinder with

$$s_{ij} = 2G\beta e_{ij}{}^E \tag{3''}$$

is considered, where β is a constant. It should be noted that when $\beta = 1.0$ the slope of J-I relation, Eq. (27), at $I = 0$ is the same as that associated with Eq. (3″); as an example, the ideal elastic-plastic relations with $\beta = 1.0$ and $\beta < 1.0$ are shown schematically by (1) and (2), respectively, in Fig. 5.

Under the same criterion of failure, the position of the elastic-plastic boundary for failure at $r = a$ is

$$r_0 = (3\beta/2)^{1/2}a \tag{32}$$

The use of Eq. (32), together with replacement of G by $G\beta$, determines the strains and stresses from Table 1; the strains are identical to Eq. (29), as expected.

The strains and stresses are computed for the same value of μ and ρ as for the nonlinear elastic solution and for β = 1.0 and 0.9, and they are shown in Figs. 6 and 7.

It appears that, for the specific values of μ and ρ chosen here, there is little difference between the stresses in the ideal elastic-plastic cylinder with $\beta = 1.0$ and the nonlinear elastic cylinder.

Figure 8 shows the critical pressure for the ideal elastic-plastic cylinder compared with that of the nonlinear elastic cylinder, Eq. (31). The ideal elastic-plastic cylinder with $\beta = 1.0$ gives a value that is about 4% above the value associated with the nonlinear elastic cylinder.

References

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